SCHUR FUNCTIONS AND INNER FUNCTIONS ON THE BIDISC

RAMLAL DEBNATH AND JAYDEB SARKAR

Abstract. We study representations of inner functions on the bidisc from a fractional linear transformation point of view. We provide sufficient conditions, in terms of colligation matrices, for the existence of two-variable inner functions. Here the sufficient conditions are not necessary in general, and we prove a weak converse for rational inner functions that admit one variable factorization.
We present a classification of de Branges-Rovnyak kernels on the bidisc (which equally works in the setting of polydisc and the open unit ball of $\mathbb{C}^n$, $n \geq 1$). We also classify, in terms of Agler kernels, two-variable Schur functions that admit one variable factor.

1. Introduction

Let $\mathbb{D}^n = \{\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|, \dots, |z_n| < 1\}$ denote the open unit polydisc and $H^\infty(\mathbb{D}^n)$ denote the Banach algebra of all bounded analytic functions on $\mathbb{D}^n$ with the uniform norm

$$\|\varphi\|_\infty := \sup\{|\varphi(\boldsymbol{z})| : \boldsymbol{z} \in \mathbb{D}^n\} \qquad (\varphi \in H^\infty(\mathbb{D}^n)).$$

A function $\varphi \in H^\infty(\mathbb{D}^n)$ is said to be *Schur function* if $\|\varphi\|_\infty \leq 1$. We denote by $\mathcal{S}(\mathbb{D}^n)$ the set of Schur functions defined on $\mathbb{D}^n$. A function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ is said to be *inner* if

$$\lim_{r \nearrow 1} |\varphi(re^{it_1}, \cdots, re^{it_n})| = |\varphi(e^{it_1}, \cdots, e^{it_n})| = 1,$$

almost everywhere on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$. For example, every rational inner function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ has the form

$$\varphi(\boldsymbol{z}) = M \frac{\overline{p(\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_n})}}{p(\boldsymbol{z})},$$

where M is a monomial and p is a polynomial with no zeros in $\mathbb{D}^n$ (see Rudin [17, Theorem 5.2.5]).

The principle aim of this paper is threefold: (1) Representations of inner functions in $\mathcal{S}(\mathbb{D}^2)$ in terms of isometric colligation operators (a certain class of 2×2 block operator matrices). (2) Classification of de Branges-Rovnyak kernels on $\mathbb{D}$ (which equally works in the setting of $\mathbb{D}^n$ and the open unit ball in $\mathbb{C}^n$, $n \geq 1$). (3) Classification, in terms of Agler kernels, of Schur functions in $\mathcal{S}(\mathbb{D}^2)$ that admit one variable factor.

To further elaborate on the main contribution of this paper, we recall some classical results. First let us recall the fractional linear transformation representations of Schur functions on

---

2010 *Mathematics Subject Classification.* 46C07, 46E22, 47A48, 47B32, 30J05, 30H10, 47A15, 47B35.
Key words and phrases. Realization formula, inner functions, Agler decompositions, Agler kernels, Schur functions, Hardy space, de Branges-Rovnyak spaces.

$\mathbb{D}$ [4, 12]. Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a function. Then $\varphi \in \mathcal{S}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{H}$ (known as *state space*) and a 2×2 block operator matrix (known as *colligation matrix/operator*)

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H},$$

such that V is an isometry(/co-isometry/unitary/contraction) and $\varphi = \tau_V$, where

$$\tau_V(z) = a + zB(I_{\mathcal{H}} - zD)^{-1}C \qquad (z \in \mathbb{D}).$$

We call τ_V the *transfer function* or the *realization function* corresponding to the colligation operator V. Moreover (see [12, Theorem 7.10, page 110, and Theorem 10.1, page 122]):

Theorem 1.1. *Let $\varphi \in \mathcal{S}(\mathbb{D})$. Then φ is inner if and only if $\varphi = \tau_V$ for some isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}),$$

with $D \in C_{0\cdot}$.

Here $C_{0\cdot}$ denotes the set of all contractions T (on Hilbert spaces) such that $T^n \to 0$ in the strong operator topology, and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ (or simply $\mathcal{B}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$) denotes the Banach space of bounded linear operators from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. We present a simpler proof of the necessary part of the above theorem at the end of this section.

Our first aim in this paper is to provide similar sufficient (as well as necessary, for reducible rational functions) conditions for a function in $\mathcal{S}(\mathbb{D}^2)$ to be inner. Our presentation here, needless to say, is based on Agler's realization formula and Agler kernels for functions in $\mathcal{S}(\mathbb{D}^2)$ [1]. Let us briefly recall the definition of kernel functions. Let $\mathcal{E}$ be a Hilbert space, and let Ω be a domain in $\mathbb{C}^n$. A function $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{E})$ is called a *kernel* (denoted by $K \geq 0$) if

$$\sum_{i,j=1}^{m} \langle K(\boldsymbol{z}_i, \boldsymbol{z}_j)\eta_j, \eta_i \rangle_{\mathcal{E}} \geq 0,$$

for all $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\} \subseteq \Omega$, $\{\eta_1, \ldots, \eta_m\} \subseteq \mathcal{E}$ and $m \geq 1$ (see the monograph by Paulsen and Raghupathi [19], or Szafraniec [20] for a rapid introduction to reproducing kernels).

Theorem 1.2 (Agler)**.** *Let $\varphi : \mathbb{D}^2 \to \mathbb{C}$ be a function. Then the following are equivalent:*

(i) $\varphi \in \mathcal{S}(\mathbb{D}^2)$.

(ii) There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary/isometric/co-isometric colligation operator

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

such that $\varphi = \tau_V$, where

$$\tau_V(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z})D)^{-1} E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z})C,$$

and $E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) = z_1 I_{\mathcal{H}_1} \oplus z_2 I_{\mathcal{H}_2}$ for all $\boldsymbol{z} \in \mathbb{D}^2$.

(iii) There exist kernels $\{K_1, K_2\}$ such that

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1 - z_1\bar{w}_1)K_1(\boldsymbol{z}, \boldsymbol{w}) + (1 - z_2\bar{w}_2)K_2(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2).$$

The kernels $\{K_1, K_2\}$ in (ii) are known as *Agler kernels* of φ, and the identity in (iii) is known as the *Agler decomposition* of φ corresponding to the Agler kernels $\{K_1, K_2\}$ [5, Section 3.1]. We also call $\mathcal{H}_1 \oplus \mathcal{H}_2$ in (ii) as a *state space* of φ.

Agler kernels were essentially introduced by Jim Agler [2] (also see [1] and [3]) in his study of Nevanlinna-Pick interpolation in the setting of bidisc. Agler kernels also play an important role (cf. [7, 15, 22]) in the delicate structure of shift-invariant subspaces of the Hardy space over the bidisc [13]. Moreover, in the case of $\mathbb{D}^n$, $n \geq 3$, Agler kernels are associated with those bounded analytic functions on $\mathbb{D}^n$ that satisfy the multivariable von Neumann inequality (see the discussion following Theorem 3.2). Needless to say, bounded analytic functions on $\mathbb{D}^n$, $n \geq 3$ satisfying the multivariable von Neumann inequality are of interest. From these points of view, the concept of Agler kernels has become an inseparable part of the modern theory of bounded analytic functions.

We now return to the topic of representations of inner functions in $\mathcal{S}(\mathbb{D}^2)$. An analog of Theorem 1.1 for inner functions in $\mathcal{S}(\mathbb{D}^2)$ seems to be a subtle and unattended problem. Here the main difficulty is to deal with the 2×2 block operator matrix $D \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, or more specifically, with the resolvent part of τ_V which involves inverse of 2×2 block operator matrix. Instead, in Theorem 2.1 we prove that a function $\varphi \in \mathcal{S}(\mathbb{D}^2)$ is inner whenever $\varphi = \tau_V$ for some isometric colligation

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $D_1, D_3 \in C_{0\cdot}$. This is the main content of Section 2.
The converse of the above fact is not true in general (see Example 5.1). However, a weak converse holds for rational inner functions that admit one variable factorization (see Theorem 5.3). These are the main content of Section 5.

Now we turn to our second goal of this paper: classification of de Branges-Rovnyak kernels on $\mathbb{D}^n$ and the open unit ball of $\mathbb{C}^n$. Here we explain the idea in the setting of operator-valued Schur functions on $\mathbb{D}$. Suppose $\Theta : \mathbb{D} \to \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ is a *Schur function*, that is, Θ is a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued analytic function on $\mathbb{D}$ and $\sup_{z \in \mathbb{D}} \|\Theta(z)\| \leq 1$ (in notation, $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$). We call the kernel

$$K_\Theta(z, w) := \frac{I - \Theta(z)\Theta(w)^*}{1 - z\overline{w}}, \qquad (z, w \in \mathbb{D}).$$

the *de Branges-Rovnyak kernel* corresponding to Θ. The classical de Branges-Rovnyak theory says that the kernel of a contractively contained shift-invariant (not necessarily closed) subspace of the Hardy space $H^2_{\mathcal{E}_*}(\mathbb{D})$ is K_Θ for some $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and Hilbert space $\mathcal{E}$. Section 3 concentrates on the following question: How can we recognize when a kernel admits a de Branges-Rovnyak kernel representation?
The following is our answer to this question (see Theorem 3.1): Let $K \geq 0$ be a $\mathcal{B}(\mathcal{E}_*)$-valued kernel (which is not a priori analytic in its first variable). Then $K = K_\Theta$ for some $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and Hilbert space $\mathcal{E}$ if and only if

$$I_{\mathcal{E}_*} - (1 - z\overline{w}) \cdot K \geq 0,$$

where $\cdot$ denotes the Hadamard product. This also covers a (variation of the) classical result due to de Branges and Rovnyak (see Theorem 3.2 and the discussion preceding it).

In the setting of Schur-Agler functions on $\mathbb{D}^n$ (see more details in Section 3), in Theorem 3.3 we prove the following: Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}^n$ (again, K is not a priori analytic in $z_1, \ldots, z_n$). Then there exist a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued Schur-Agler function Θ (in notation, $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$) such that

$$K = K_\Theta,$$

where

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) := \frac{I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{\prod\limits_{i=1}^{n}(1 - z_i \bar{w}_i)} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

if and only if there exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \ldots, K_n$ (we call it *Agler kernels* of φ) on $\mathbb{D}^n$ such that

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i}(1 - z_j \bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

and

$$I_{\mathcal{E}_*} - \Big(\prod_{i=1}^{n}(1 - z_i \bar{w}_i)\Big) \cdot K \geq 0.$$

An analogous but somewhat simpler statement also holds in the setting of multipliers of Drury-Arveson space (see Theorem 3.4).

The final goal of this paper is to describe those two-variable Schur functions that admit one variable Schur factor. This is the main content of Section 4. More specifically (see Theorem 4.1): Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ and suppose $\varphi(\mathbf{0}) \neq 0$, where $\mathbf{0} = (0, 0)$ (see Remark 4.3 on the assumption $\varphi(\mathbf{0}) \neq 0$). The following assertions are equivalent:

(1) There exist φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ such that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$.

(2) There exist Agler kernels $\{K_1, K_2\}$ of φ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(3) There exist Agler kernels $\{L_1, L_2\}$ of φ such that all the functions in $\mathcal{H}_{L_1}$ depends only on z_1, and $\varphi(\mathbf{0}) f(\cdot, 0) = \varphi(\cdot, 0)\, f(\mathbf{0})$, $f \in \mathcal{H}_{L_2}$ (here $\mathcal{H}_K$ refers to the reproducing kernel Hilbert space corresponding to a kernel K).

(4) $\varphi = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $\varphi(\mathbf{0}) D_2 = C_1 B_2$.

We remark that, given the importance of the rich structure, inner functions on the bidisc have been considered in many occasions previously in different contexts (cf. [7, 22]). We also refer to [6, 15, 21] and the references therein for the recent and more modern development of Schur functions, Agler kernels, and transfer function realizations.

It is worthwhile to point out that our main motivation of colligation matrices, as in part (4) above and the one following Theorem 1.2, comes from the recent paper [10]. That said, the present paper is intended as a follow-up, to some extent, to [10] on one hand, and is also designed to be self-contained on the other.

We end this section with a simple proof of the necessary part of Theorem 1.1. The idea of the proof (as also hinted at the final paragraph of this section) is the same as that used in proving analytic representations of commutators of shift operators [16, Theorem 2]. Let $\varphi \in \mathcal{S}(\mathbb{D})$ be an inner function, and suppose

$$\varphi = \sum_{m=0}^{\infty} a_m z^m,$$

the power series representation of φ on $\mathbb{D}$. Consider M_φ as an isometric multiplier on $H^2(\mathbb{D})$ (that is, $\|M_\varphi f\| = \|f\|$ for all $f \in H^2(\mathbb{D})$), and set $\mathcal{Q}_\varphi = H^2(\mathbb{D}) \ominus \varphi H^2(\mathbb{D})$ (that is, $\mathcal{Q}_\varphi$ is the orthogonal complement of $\varphi H^2(\mathbb{D})$ in $H^2(\mathbb{D})$). We clearly have

$$a_m = P_{\mathbb{C}} M_z^{*m} M_\varphi|_{\mathbb{C}} \qquad (m \geq 0),$$

where $P_{\mathbb{C}}$ denotes the orthogonal projection onto the space of constant functions in $H^2(\mathbb{D})$. Note that $M_\varphi|_{\mathbb{C}} = \varphi$. Since $M_z^* \mathcal{Q}_\varphi \subseteq \mathcal{Q}_\varphi$ and $M_z^* \varphi \in \mathcal{Q}_\varphi$ (indeed, $\langle M_z^* \varphi, \varphi f\rangle = \langle M_z^* 1, f\rangle = 0$ for all $f \in H^2(\mathbb{D})$), it follows that

$$\varphi(w) = P_{\mathbb{C}} M_\varphi|_{\mathbb{C}} + w P_{\mathbb{C}}|_{\mathcal{Q}_\varphi} (I_{\mathcal{Q}_\varphi} - w M_z^*|_{\mathcal{Q}_\varphi})^{-1} M_z^* M_\varphi|_{\mathbb{C}} \qquad (w \in \mathbb{D}). \tag{1.1}$$

Clearly

$$V = \begin{bmatrix} \varphi(0) & P_{\mathbb{C}}|_{\mathcal{Q}_\varphi} \\ M_z^* M_\varphi|_{\mathbb{C}} & M_z^*|_{\mathcal{Q}_\varphi} \end{bmatrix},$$

defines a unitary colligation operator on $\mathbb{C} \oplus \mathcal{Q}_\varphi$. And, of course, we have $M_z^*|_{\mathcal{Q}_\varphi} \in C_{0\cdot}$ and $\varphi = \tau_V$.

Note that the representation of φ in (1.1) reduces to a more compact form as

$$\varphi(w) = P_{\mathbb{C}} (I_{H^2(\mathbb{D})} - w M_z^*)^{-1} M_\varphi|_{\mathbb{C}} \qquad (w \in \mathbb{D}).$$

This formula is comparable to the representation of commutators in the statement of Theorem [16, Theorem 2]. In other words, the above result also follows from Theorem [16, Theorem 2]. However, proof of the sufficient part (as in [12]) of Theorem 1.1 remains to be involved.

2. Inner Functions and Realizations

Our purpose here is to prove an analogous statement of the sufficient part of Theorem 1.1. We will again return to this topic in Section 5 with some counterexamples and a weak converse.

It will be convenient, to begin with, some terminology and basic observations. The following construction also could be of some independent interest. We write $\oplus l^2 = l^2 \oplus l^2 \oplus \cdots$, that is

$$\oplus l^2 = \Big\{ \{a_{ij}\} := \Big\{ \{\{a_{0j}\}_{j\geq 0}, \{a_{1j}\}_{j\geq 0}, \{a_{2j}\}_{j\geq 0}, \ldots\} : \sum_{i,j=0}^{\infty} |a_{ij}|^2 < \infty \Big\}.$$

One can easily verify that

$$\tau(\{a_{ij}\}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} z_1^i z_2^j,$$

defines a unitary $\tau : \oplus l^2 \to H^2(\mathbb{D}^2)$, and $M_{z_1}\tau = \tau S$, where S denotes the *shift* on $\oplus l^2$, that is

$$S\Big(\{a_{ij}\}\Big) = \Big\{\{0\}, \{a_{0j}\}_{j\geq 0}, \{a_{1j}\}_{j\geq 0}, \dots\Big\}.$$

Here $\{0\} \in l^2$ is the zero sequence. Now, let $\varphi = \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} \varphi_{ij} z_2^j) z_1^i \in H^\infty(\mathbb{D}^2)$. We define the *block Toeplitz operator* with symbol φ to be the bounded linear operator T_φ on $\oplus l^2$ defined by

$$\Big(T_\varphi\Big(\{a_{ij}\}\Big)\Big)_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} \varphi_{i-k,j-l} a_{kl} \qquad (i, j \geq 0),$$

which in matrix notation becomes

$$T_\varphi = \begin{bmatrix} \Phi_0 & 0 & 0 & 0 & \cdots \\ \Phi_1 & \Phi_0 & 0 & 0 & \cdots \\ \Phi_2 & \Phi_1 & \Phi_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\Phi_k = \begin{bmatrix} \varphi_{k0} & 0 & 0 & 0 & \cdots \\ \varphi_{k1} & \varphi_{k0} & 0 & 0 & \cdots \\ \varphi_{k2} & \varphi_{k1} & \varphi_{k0} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is a Toeplitz operator on l^2 for all $k \geq 0$. More specifically, we have

$$M_\varphi \tau = \tau T_\varphi \qquad (\varphi \in H^\infty(\mathbb{D}^2)),$$

where M_φ denotes the multiplication operator on $H^2(\mathbb{D}^2)$ with analytic symbol φ, that is, $M_\varphi f = \varphi f$ for all $f \in H^2(\mathbb{D}^2)$. Indeed, if $\varphi = \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} \varphi_{ij} z_2^j) z_1^i$ and $\{a_{ij}\} \in \oplus l^2$, then

$$\begin{aligned}
M_\varphi \tau(\{a_{ij}\}) &= \left(\sum_{i,j=0}^{\infty} \varphi_{ij} z_1^i z_2^j\right)\left(\sum_{k,l=0}^{\infty} a_{kl} z_1^k z_2^l\right) \\
&= \sum_{j,l=0}^{\infty} \sum_{i,k=0}^{\infty} \varphi_{ij} a_{kl} z_1^{i+k} z_2^{l+j} \\
&= \sum_{j,l=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} \varphi_{i-k,j} a_{kl} \right) z_1^i \right\} z_2^{l+j} \\
&= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left\{ \sum_{j,l=0}^{\infty} \varphi_{i-k,j} a_{kl} z_2^{l+j} \right\} z_1^i \\
&= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left\{ \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} \varphi_{i-k,j-l} a_{kl} \right) z_2^j \right\} z_1^i \\
&= \sum_{i,j=0}^{\infty} \left(\sum_{k,l=0}^{i,j} \varphi_{i-k,j-l} a_{kl} \right) z_1^i z_2^j,
\end{aligned}$$

and hence $M_\varphi \tau(\{a_{ij}\}) = \tau T_\varphi(\{a_{ij}\})$. In particular, we have

$$T_{z_2} = \begin{bmatrix} S_{l^2} & 0 & 0 & 0 & \cdots \\ 0 & S_{l^2} & 0 & 0 & \cdots \\ 0 & 0 & S_{l^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where S_{l^2} denotes the shift on l^2, that is, $S_{l^2}(\{a_0, a_1, \ldots\}) = \{0, a_0, a_1, \ldots\}$ for all $\{a_m\}_{m\geq 0} \in l^2$. Continuing with the above notation, we set

$$Y_0 = \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix}, \quad \text{and} \quad Y_j = S^j Y_0, \tag{2.1}$$

for all $j \geq 1$. Then $T_\varphi = \begin{bmatrix} Y_0 & Y_1 & Y_2 & \ldots \end{bmatrix}$. Since $T_\varphi^* T_\varphi = (Y_i^* Y_j)$, it follows that M_φ on $H^2(\mathbb{D}^2)$ is an isometry if and only if T_φ on $\oplus l^2$ is an isometry, which is also equivalent to

$$Y_i^* Y_j = \delta_{ij} I_{l^2}. \tag{2.2}$$

We are now ready to present the main theorem of this section.

Theorem 2.1. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$. If $\varphi = \tau_V$ for some isometric colligation*

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2),$$

with $D_1, D_3 \in C_{0\cdot}$, then φ is an inner function.

Proof. Since $\varphi = \tau_V$, and

$$\tau_V(\boldsymbol{z}) = a + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \right)^{-1} E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

we have

$$\varphi(\boldsymbol{z}) = a + \sum_{i=1}^{\infty} B_1 D_1^{i-1} C_1 z_1^i + \sum_{j=1}^{\infty} B_2 D_3^{j-1} C_2 z_2^j + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B_1 D_1^{i-1} D_2 D_3^{j-1} C_2 z_1^i z_2^j,$$

for all $\boldsymbol{z} \in \mathbb{D}^2$. Under the same notations preceding the statement, we set

$$\Phi_0 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots \\ B_2 C_2 & a & 0 & 0 & \cdots \\ B_2 D_3 C_2 & B_2 C_2 & a & 0 & \cdots \\ B_2 D_3^2 C_2 & B_2 D_3 C_2 & B_2 C_2 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$\Phi_j = \begin{bmatrix} B_1 D_1^{j-1} C_1 & 0 & 0 & 0 & \cdots \\ B_1 D_1^{j-1} D_2 C_2 & B_1 D_1^{j-1} C_1 & 0 & 0 & \cdots \\ B_1 D_1^{j-1} D_2 D_3 C_2 & B_1 D_1^{j-1} D_2 C_2 & B_1 D_1^{j-1} C_1 & 0 & \cdots \\ B_1 D_1^{j-1} D_2 D_3^2 C_2 & B_1 D_1^{j-1} D_2 D_3 C_2 & B_1 D_1^{j-1} D_2 C_2 & B_1 D_1^{j-1} C_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

for $j \geq 1$. We first claim that $Y_0 = \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix}$ is an isometry. In fact, since $Y_0^* Y_0 = \sum_{m=0}^{\infty} \Phi_m^* \Phi_m$, there exists a sequence of scalars $\{y_m\}_{m \geq 0}$ such that

$$Y_0^* Y_0 = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots \\ \overline{y_1} & y_0 & y_1 & \cdots \\ \overline{y_2} & \overline{y_1} & y_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We need to show that $y_0 = 1$ and $y_k = 0$ for all $k \geq 1$. Note that

$$y_0 = |a|^2 + C_1^* \left(\sum_{j=0}^{\infty} D_1^{*j} B_1^* B_1 D_1^j \right) C_1 + C_2^* \left[\sum_{k=0}^{\infty} D_3^{*k} \left\{ B_2^* B_2 + D_2^* \left(\sum_{l \geq 0} D_1^{*l} B_1^* B_1 D_1^l \right) D_2 \right\} D_3^k \right] C_2.$$

Since $V^*V = I$, it follows that

$$\begin{bmatrix} |a|^2 + C_1^*C_1 + C_2^*C_2 & \bar{a}B_1 + C_1^*D_1 & \bar{a}B_2 + C_1^*D_2 + C_2^*D_3 \\ aB_1^* + D_1^*C_1 & B_1^*B_1 + D_1^*D_1 & B_1^*B_2 + D_1^*D_2 \\ aB_2^* + D_2^*C_1 + D_3^*C_2 & B_2^*B_1 + D_2^*D_1 & B_2^*B_2 + D_2^*D_2 + D_3^*D_3 \end{bmatrix} = I. \tag{2.3}$$

In particular

$$\begin{aligned} I &= B_1^*B_1 + D_1^*D_1 \\ &= B_1^*B_1 + D_1^*(B_1^*B_1 + D_1^*D_1)D_1 \\ &= B_1^*B_1 + D_1^*B_1^*B_1D_1 + D_1^{*2}D_1^2, \end{aligned}$$

and hence $I = \sum_{j=0}^{m} D_1^{*j}(B_1^*B_1)D_1^j + D_1^{*(m+1)}D_1^{(m+1)}$ for all $m \geq 1$. Using the fact that $D_1 \in C_{0\cdot}$, we have

$$\sum_{j=0}^{\infty} D_1^{*j}(B_1^*B_1)D_1^j = I, \tag{2.4}$$

in the strong operator topology. Similarly, $B_2^*B_2 + D_2^*D_2 + D_3^*D_3 = I$ and $D_3 \in C_{0\cdot}$ implies that

$$\sum_{j=0}^{\infty} D_3^{*j}(B_2^*B_2 + D_2^*D_2)D_3^j = I, \tag{2.5}$$

in the strong operator topology. This with the condition $|a|^2 + C_1^*C_1 + C_2^*C_2 = 1$ in (2.3) implies that

$$y_0 = |a|^2 + C_1^*C_1 + C_2^*C_2 = 1.$$

Next we consider

$$y_1 = aC_2^*B_2^* + C_2^*D_2^* \left(\sum_{j=0}^{\infty} D_1^{*j} B_1^* B_1 D_1^j \right) C_1 + C_2^*D_3^* \left[\sum_{k=0}^{\infty} D_3^{*k} \left\{ B_2^* B_2 + D_2^* \left(\sum_{l=0}^{\infty} D_1^{*l} B_1^* B_1 D_1^l \right) D_2 \right\} D_3^k \right] C_2 \, .$$

Thus by (2.4) and (2.5), it follows that

$$y_1 = aC_2^*B_2^* + C_2^*D_2^*C_1 + C_2^*D_3^*C_2 = C_2^*(aB_2^* + D_2^*C_1 + D_3^*C_2) = 0,$$

as $aB_2^* + D_2^* C_1 + D_3^* C_2 = 0$ follows from (2.3). Similarly

$$y_j = C_2^* D_3^{*j}(aB_2^* + D_2^* C_1 + D_3^* C_2) = 0,$$

for all $j \geq 2$. This proves that Y_0 is an isometry.
Since the shift S on $\oplus l^2$ is an isometry, $Y_j := S^j Y_0$, $j \geq 1$, is also an isometry (see the construction in (2.1)). Our final goal is to prove that $T_\varphi := \begin{bmatrix} Y_0 & Y_1 & Y_2 & \dots \end{bmatrix}$ is an isometry, or equivalently, by virtue of (2.2) and $Y_m^* Y_m = I$ for all $m \geq 0$,

$$Y_p^* Y_q = 0 \qquad (p > q \geq 0).$$

Since $Y_p^* Y_q = Y_0^* S^{*p} S^q Y_0 = Y_0^* S^{*(p-q)} Y_0$ for all $p > q \geq 0$, it actually suffices to check that

$$Y_0^* S^{*(j+1)} Y_0 = 0 \qquad\qquad (j \geq 0).$$

So we fix $j \geq 0$ and observe

$$S^j Y_0 = \begin{bmatrix} \underbrace{0 \ \dots \ 0}_{(j+1)} & \Phi_0 & \Phi_1 & \Phi_2 & \dots \end{bmatrix}^t .$$

Hence

$$Y_0^* S^{*(j+1)} Y_0 = \Phi_0^* \Phi_{j+1} + \Phi_1^* \Phi_{j+2} + \cdots .$$

Therefore there exists a sequence $\{c_m\}_{m \in \mathbb{Z}}$ such that

$$Y_0^* S^{*(j+1)} Y_0 = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots \\ c_{-1} & c_0 & c_1 & \cdots \\ c_{-2} & c_{-1} & c_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is suffices to prove that $c_k = 0$ for all $k \in \mathbb{Z}$. A simple calculation shows that

$$c_0 = (\bar{a}B_1 + C_1^* D_1) D_1^{j+1} C_1 + C_2^* \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_1 \right\} D_1^{j+1} D_2 D_3^m \right] C_2.$$

By (2.3), $\bar{a}B_1 + C_1^* D_1 = 0$ and $B_2^* B_1 + D_2^* D_1 = 0$, and hence $c_0 = 0$. Now let $k > 0$. Then

$$c_k = C_2^* D_3^{*(k-1)} (B_2^* B_1 + D_2^* D_1) D_1^{j+1} C_1 + C_2^* D_3^{*k} \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_1 \right\} D_1^{j+1} D_2 D_3^m \right] C_2,$$

and hence $c_k = 0$. Finally, since

$$c_{-k} = (\overline{a}B_1 + C_1^* D_1) D_1^{j+1} D_2 D_3^{k-1} C_2 + C_2^* \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_{1,1} \right\} D_1^{j+1} D_2 D_3^m \right] D_3^k C_2,$$

it again follows that $c_{-k} = 0$. This implies that T_φ or, equivalently, M_φ is an isometry, and completes the proof. ∎

Remark 2.2. Let $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H})$ be an isometric colligation, and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ for some Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose $D := P_{\mathcal{H}} V|_{\mathcal{H}}$ and suppose that $D\mathcal{H}_1 \subseteq \mathcal{H}_1$. Set

$$D = \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2).$$

It is easy to see that if $D \in C_{0\cdot}$, then D_1 and D_3 are also in $C_{0\cdot}$. Consequently, Theorem 2.1 also holds for those Schur functions φ such that $\varphi = \tau_V$ with V as above. Of course, if D_1 and D_3 are in $C_{0\cdot}$, then D is not necessarily in $C_{0\cdot}$.

3. de Branges-Rovnyak kernels

The goal of this section is to study de Branges-Rovnyak kernels on $\mathbb{D}^n$ and the open unit ball of $\mathbb{C}^n$, $n \geq 1$. Specifically, we seek characterizations of analytic kernels that admit certain factorizations involving Schur(-Agler) functions. Our investigation is partly motivated by a classical result of de Branges and Rovnyak (see the paragraph preceding Theorem 3.2 for more details).

We start with the unit disc case. Let $\mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ denote the set of all $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued analytic functions Θ on $\mathbb{D}$ such that $\sup_{z \in \mathbb{D}} \|\Theta(z)\| \leq 1$. Such functions are called *operator-valued Schur functions.*

A kernel $K : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ is a *de Branges-Rovnyak kernel* if there exists $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that

$$K(z, w) = K_\Theta(z, w) := \frac{I - \Theta(z)\Theta(w)^*}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}).$$

Note that if $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$, then $M_\Theta : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}_*}(\mathbb{D})$ is a contraction. This is clearly equivalent to the condition that $K_\Theta \geq 0$.

In the following, we characterize de Branges-Rovnyak kernels defined on the disc $\mathbb{D}$. The proof uses the commonly used "lurking-isometry" techniques. Therefore, our proof is fairly standard and, perhaps, it can also be achieved from existing results of Schur(-Agler) functions [4]. Note also that the theorem below does not assume a priori that K is analytic in its first variable.

Theorem 3.1. *Let $K : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}$. Then $K = K_\Theta$ for some $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and Hilbert space $\mathcal{E}_*$ if and only if*

$$I_{\mathcal{E}_*} - (1 - z\bar{w}) \cdot K \geq 0.$$

Proof. If $K = K_\Theta$, then

$$I_{\mathcal{E}_*} - (1 - z\bar{w})K(z, w) = \Theta(z)\Theta(w)^* \geq 0 \qquad (z, w \in \mathbb{D}).$$

Conversely, if $I_{\mathcal{E}_*} - (1 - z\overline{w}) \cdot K \geq 0$, then there exist a Hilbert space $\mathcal{F}$ and a function (a priori not necessarily analytic) $F : \mathbb{D} \to \mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ such that

$$I_{\mathcal{E}} - (1 - z\bar{w})K(z, w) = F(z)F(w)^* \qquad (z, w \in \mathbb{D}).$$

Clearly, F is a contractive function on $\mathbb{D}$. Again, since $K \geq 0$, there exist a Hilbert space $\mathcal{G}$ and a function $G : \mathbb{D} \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that $K(z, w) = G(z)G(w)^*$, $z, w \in \mathbb{D}$. Then

$$I_{\mathcal{E}_*} - G(z)G(w)^* + z\bar{w}G(z)G(w)^* = F(z)F(w)^*,$$

and hence

$$I_{\mathcal{E}_*} + z\bar{w}G(z)G(w)^* = G(z)G(w)^* + F(z)F(w)^*,$$

for all $z, w \in \mathbb{D}$. Therefore

$$V : \begin{bmatrix} I_{\mathcal{E}_*} \\ \bar{w}G(w)^* \end{bmatrix} \eta \mapsto \begin{bmatrix} F(w)^* \\ G(w)^* \end{bmatrix} \eta \qquad (w \in \mathbb{D}, \eta \in \mathcal{E}_*),$$

defines an isometry from a subspace of $\mathcal{E}_* \oplus \mathcal{G}$ to $\mathcal{F} \oplus \mathcal{G}$. Then, adding an infinite-dimensional summand to $\mathcal{G}$ if necessary, V can then be extended to an isometry, denoted by V again, from $\mathcal{E}_* \oplus \mathcal{G}$ to $\mathcal{F} \oplus \mathcal{G}$. Set

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{E}_* \oplus \mathcal{G} \to \mathcal{F} \oplus \mathcal{G}.$$

Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \eta \\ \bar{w}G(w)^*\eta \end{bmatrix} = \begin{bmatrix} F(w)^*\eta \\ G(w)^*\eta \end{bmatrix},$$

for all $\eta \in \mathcal{E}$ and $w \in \mathbb{D}$, which implies that

$$A + \bar{w}BG(w)^* = F(w)^* \text{ and } C + \bar{w}DG(w)^* = G(w)^*,$$

for all $w \in \mathbb{D}$. The latter equality implies that $G(w)^* = (I - \overline{w}D)^{-1}C$, and hence, the first equality yields

$$F(w)^* = A + \overline{w}B(I - \overline{w}D)^{-1}C,$$

for all $w \in \mathbb{D}$. Hence

$$F(z) = A^* + zC^*(I - zD^*)^{-1}B^* \qquad (z \in \mathbb{D}),$$

that is, $F = \tau_{V^*}$ is analytic on $\mathbb{D}$ and bounded by 1, where

$$V^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

is a co-isometric colligation. Consequently, $\Theta := F \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{F}, \mathcal{E}_*))$, and hence

$$I_{\mathcal{E}_*} - (1 - z\bar{w})K(z, w) = \Theta(z)\Theta(w)^*,$$

that is, $K(z, w) = \dfrac{I_{\mathcal{E}_*} - \Theta(z)\Theta(w)^*}{1 - z\bar{w}}$ for all $z, w \in \mathbb{D}$. This completes the proof. ∎

We denote by $\mathbb{S}_n$ the Szegö kernel on $\mathbb{D}^n$, that is

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \frac{1}{1 - z_i\bar{w}_i} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Also we denote $\mathbb{S}_1$ simply by $\mathbb{S}$. The following is a variation of a result due to de Branges and Rovnyak [8, 9]. Also, we refer the reader to the classic Sz.-Nagy and Foias [18, Section 8, page 231] for detailed proof and some historical notes. The proof below follows the proof of the previous theorem. Again, a priori we do not assume (in contrast to Sz.-Nagy and Foias) that K is analytic in its first variable.

Theorem 3.2. *Let $K : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E}_*)$ be a kernel. Then*

$$0 \le K \le \mathbb{S} \text{ and } \mathbb{S}^{-1} \cdot K \ge 0,$$

if and only if there exist a Hilbert space $\mathcal{E}$ and an operator-valued Schur function $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that

$$K(z, w) = \frac{\Theta(z)\Theta(w)^*}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}).$$

Proof. Suppose $0 \le K \le \mathbb{S}$ and $\mathbb{S}^{-1} \cdot K \ge 0$. Now $0 \le K \le \mathbb{S}$ implies that

$$\frac{1}{1 - z\bar{w}} I - K(z, w) \ge 0.$$

As in the proof of the previous theorem, there exist a Hilbert space $\mathcal{F}$ and a function $G : \mathbb{D} \to \mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ such that

$$I - (1 - z\bar{w})K(z, w) = (1 - z\bar{w})G(z)G(w)^*.$$

Again, since $\mathbb{S}^{-1} \cdot K \ge 0$, there exist a Hilbert space $\mathcal{G}$ and a function $F : \mathbb{D} \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that

$$I - F(z)F(w)^* = (1 - z\bar{w})G(z)G(w)^*$$

The remaining argument is similar to that of the proof of the previous theorem. ■

Now we recall the definition of Schur-Agler functions. Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces. The *Schur-Agler class* $\mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ [1] consists of $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued analytic functions φ on $\mathbb{D}^n$ such that φ satisfies the n-variables von Neumann inequality

$$\|\varphi(T_1, \ldots, T_n)\|_{\mathcal{B}(\mathcal{H})} \le 1,$$

for any n-tuples of commuting strict contractions on a Hilbert space $\mathcal{H}$. Here

$$\varphi(T_1, \ldots, T_n) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \varphi_{\boldsymbol{k}} \otimes T^{\boldsymbol{k}},$$

where $\varphi = \sum\limits_{\boldsymbol{k} \in \mathbb{Z}_+^n} \varphi_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$, $\varphi_{\boldsymbol{k}} \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, and $T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n}$ for all $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$. The elements of $\mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ are called *Schur-Agler functions*. The following result is due to Agler [1] (also see Theorem 1.2):
Given a function $\Theta : \mathbb{D}^n \to \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, the following are equivalent:
(i) $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$.
(ii) There exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \ldots, K_n$ (known as *Agler kernels*) on $\mathbb{D}^n$ such that

$$I_{\mathcal{E}_*} - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^* = \sum_{i=1}^{n} (1 - z_i \bar{w}_i) K_i(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

We now turn to de Branges-Rovnyak kernels on $\mathbb{D}^n$. Suppose $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. Since M_Θ is a contraction from $H^2_{\mathcal{E}}(\mathbb{D}^n)$ into $H^2_{\mathcal{E}_*}(\mathbb{D}^n)$, it is easy to check (as also pointed out earlier) that $K_\Theta \ge 0$, where

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w})^{-1}(I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Here we say that K_Θ is a ($\mathcal{B}(\mathcal{E}_*)$-valued) de Branges-Rovnyak kernel on $\mathbb{D}^n$. In the following, we do not assume a priori that K is analytic in $z_1, \dots, z_n$.

Theorem 3.3. *Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}^n$. Then $K = K_\Theta$ for some Schur-Agler function $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and a Hilbert space $\mathcal{E}$ if and only if there exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \dots, K_n$ on $\mathbb{D}^n$ such that*

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i}(1 - z_j \bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}),$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$, and $I_{\mathcal{E}_*} - \mathbb{S}_n^{-1} \cdot K \geq 0$.

Proof. The "only if" part of this statement is easy, and the proof of the "if" part is similar to the proof of Theorem 3.1. We give only a sketch: Suppose $K_1, \dots, K_n$ are $\mathcal{B}(\mathcal{E}_*)$-valued kernels on $\mathbb{D}^n$, $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$, and suppose

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i}(1 - z_j \bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}).$$

Then

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} (1 - z_i \bar{w}_i) K_i(\boldsymbol{z}, \boldsymbol{w}).$$

Since $I_{\mathcal{E}_*} - \mathbb{S}_n^{-1} \cdot K \geq 0$, there exist a Hilbert space $\mathcal{G}$ and a function $G : \mathbb{D}^n \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that

$$I_{\mathcal{E}} - \mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) K(\boldsymbol{z}, \boldsymbol{w}) = G(\boldsymbol{z}) G(\boldsymbol{w})^*.$$

Again, since $K_i \geq 0$, there exist Hilbert spaces $\mathcal{F}_1, \dots, \mathcal{F}_n$, and functions $F_i : \mathbb{D}^n \to \mathcal{B}(\mathcal{F}_i, \mathcal{E}_*)$, $i = 1, \dots, n$, such that $K_i(\boldsymbol{z}, \boldsymbol{w}) = F_i(\boldsymbol{z}) F_i(\boldsymbol{w})^*$ for all $i = 1, \dots, n$. Hence

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i}^{n} (1 - z_i \bar{w}_i) F_i(\boldsymbol{z}) F_i(\boldsymbol{w})^*,$$

which implies

$$I_{\mathcal{E}_*} + \sum_{i=1}^{n} z_i \bar{w}_i F_i(\boldsymbol{z}) F_i(\boldsymbol{w})^* = G(\boldsymbol{z}) G(\boldsymbol{w})^* + \sum_{i=1}^{n} F_i(\boldsymbol{z}) F_i(\boldsymbol{w})^*,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$. Now one can proceed with the lurking-isometry method, as in the proof of Theorem 3.1, to complete the proof of the theorem. ∎

An analogous statement also holds in the case of multipliers of *Drury-Arveson space* H_n^2. We recall that H_n^2 is the reproducing kernel Hilbert space corresponding to the kernel

$$S(\boldsymbol{z}, \boldsymbol{w}) := \frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

where $\mathbb{B}^n = \{\boldsymbol{z} \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1\}$ is the open unit ball of $\mathbb{C}^n$, and $\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \sum_{i=1}^{n} z_i \bar{w}_i$. Given Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued Drury-Arveson multiplier space is defined by

$$\mathcal{M}_n(\mathcal{E}, \mathcal{E}_*) = \{\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{E}_*) : \Theta(H_n^2 \otimes \mathcal{E}) \subseteq H_n^2 \otimes \mathcal{E}_*\}.$$

Here $\mathcal{M}_n(\mathcal{E}, \mathcal{E}_*)$ is a Banach space equipped with the norm $\|\Theta\|_{\mathcal{M}_n(\mathcal{E},\mathcal{E}_*)} = \|M_\Theta\|$ (the operator norm of M_Θ). In this setting, the de Branges-Rovnyak kernel K_Θ corresponding to $\Theta \in \mathcal{M}_d(\mathcal{E}, \mathcal{E}_*)$ is defined by

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) = \frac{I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z}, \boldsymbol{w}\rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

The proof of the following theorem is completely analogous to the proof of Theorems 3.1 and 3.3. We leave details to the reader.

Theorem 3.4. *Let $\mathcal{E}_*$ be a Hilbert space and $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{B}(\mathcal{E})$ be a kernel. Then $K = K_\Theta$ for some $\Theta \in \mathcal{M}_n(\mathcal{E}, \mathcal{E}_*)$ and Hilbert space $\mathcal{E}$ if and only if*

$$I_{\mathcal{E}_*} - (1 - \langle \boldsymbol{z}, \boldsymbol{w}\rangle) \cdot K(\boldsymbol{z}, \boldsymbol{w}) \geq 0.$$

In the above theorem, we do not assume a priori that K is analytic in $z_1, \ldots, z_n$.

4. Agler Kernels and Factorizations

In this section we investigate factorizations of two-variable Schur functions in terms of Agler kernels. We shall be particularly interested in the case of one variable factors and Agler kernels of functions in $\mathcal{S}(\mathbb{D}^2)$.

Here and in what follows, $\mathcal{H}_K$ will denote the reproducing kernel Hilbert space corresponding to the kernel K. Moreover, if $K : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$, then $K(\cdot, \boldsymbol{w}) \in \mathcal{H}_K$ will denote the kernel function at $\boldsymbol{w} \in \mathbb{D}^2$, that is

$$\Big(K(\cdot, \boldsymbol{w})\Big)(\boldsymbol{z}) = K(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z} \in \mathbb{D}^2),$$

and

$$f(\boldsymbol{w}) = \langle f, K(\cdot, \boldsymbol{w})\rangle_{\mathcal{H}_K},$$

for all $f \in \mathcal{H}_K$ and $\boldsymbol{w} \in \mathbb{D}^2$. For notational convenience we write $\mathbf{0} = (0, 0)$.

We are now ready for the main result of this section (see Remark 4.3 on the assumption $\varphi(\mathbf{0}) \neq 0$):

Theorem 4.1. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ and suppose $\varphi(\mathbf{0}) \neq 0$. The following assertions are equivalent:*
(1) There exist φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ such that

$$\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2) \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

(2) There exist Agler kernels $\{K_1, K_2\}$ of φ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(3) There exist Agler kernels $\{L_1, L_2\}$ of φ such that all the functions in $\mathcal{H}_{L_1}$ depends only on z_1, and

$$\varphi(\mathbf{0}) f(\cdot, 0) = \varphi(\cdot, 0)\, f(\mathbf{0}) \qquad (f \in \mathcal{H}_{L_2}).$$

(4) $\varphi = \tau_V$ *for some co-isometric colligation*

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $\varphi(\mathbf{0}) D_2 = C_1 B_2$.

Proof. Suppose first that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$. Then

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = 1 - \varphi_1(z_1)\overline{\varphi_1(w_1)} + \varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\,\overline{\varphi_1(w_1)},$$

and hence

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1 - z_1\bar{w}_1)K_1(\boldsymbol{z}, \boldsymbol{w}) + (1 - z_2\bar{w}_2)K_2(\boldsymbol{z}, \boldsymbol{w}),$$

where

$$K_1(\boldsymbol{z}, \boldsymbol{w}) = \frac{1 - \varphi_1(z_1)\overline{\varphi_1(w_1)}}{1 - z_1\bar{w}_1} \quad \text{and} \quad K_2(\boldsymbol{z}, \boldsymbol{w}) = \frac{\varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\,\overline{\varphi_1(w_1)}}{1 - z_2\bar{w}_2},$$

and $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$. Then $\{K_1, K_2\}$ are Agler kernels of φ and satisfies the conditions of (2). This proves (1)$\Rightarrow$(2).
(2)$\Rightarrow$ (3): Set $L_i = K_i$, $i = 1, 2$, and suppose $f \in \mathcal{H}_{K_1}$. Since

$$\boldsymbol{w} \mapsto f(w) = \langle f, L_1(\cdot, \boldsymbol{w})\rangle = \langle f, K_1(\cdot, \boldsymbol{w})\rangle,$$

and K_1 depends only on z_1 and $\bar{w}_1$, it follows that all the functions in $\mathcal{H}_{L_1}$ depends only on z_1. On the other hand, if $f \in \mathcal{H}_{L_2}$, then

$$\begin{aligned} \varphi(\mathbf{0}) f(w_1, 0) &= \langle f, \overline{\varphi(\mathbf{0})} L_2(\cdot, (w_1, 0))\rangle_{\mathcal{H}_{L_2}} \\ &= \langle f, \overline{\varphi(\mathbf{0})} K_2(\cdot, (w_1, 0))\rangle_{\mathcal{H}_{K_2}} \\ &= \langle f, \overline{\varphi(w_1, 0)} K_2(\cdot, (\mathbf{0}))\rangle_{\mathcal{H}_{K_2}} \\ &= \varphi(w_1, 0)\langle f, K_2(\cdot, (\mathbf{0}))\rangle_{\mathcal{H}_{K_2}}, \end{aligned}$$

and hence $\varphi(\mathbf{0}) f(w_1, 0) = \varphi(w_1, 0) f(\mathbf{0})$ for all $w_1 \in \mathbb{D}$.
(3)$\Rightarrow$ (2): This is just the reverse of the argument in the above proof.
(2)$\Rightarrow$ (4): Suppose $\{K_1, K_2\}$ are Agler kernels of φ, and suppose that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}). \tag{4.1}$$

Now

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1 - z_1\bar{w}_1)\langle K_1(\cdot, \boldsymbol{w}), K_1(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + (1 - z_2\bar{w}_2)\langle K_2(\cdot, \boldsymbol{w}), K_2(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_2}},$$

implies that

$$\begin{aligned} 1 + z_1\bar{w}_1\langle K_1(\cdot, \boldsymbol{w}), K_1(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + z_2\bar{w}_2\langle K_2(\cdot, \boldsymbol{w}), K_2(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_2}} &= \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} \\ &\quad + \langle K_1(\cdot, \boldsymbol{w}), K_1(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + \langle K_2(\cdot, \boldsymbol{w}), K_2(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_2}}, \end{aligned}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$. Therefore

$$V : \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \mapsto \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

defines an isometry from $\mathcal{D}$ onto $\mathcal{R}$, where

$$\mathcal{D} = \overline{\text{span}} \left\{ \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2},$$

and

$$\mathcal{R} = \overline{\text{span}} \left\{ \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}.$$

Note that

$$\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2} = \overline{\text{span}} \left\{ \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\}.$$

Indeed, if

$$\begin{bmatrix} f \\ g \end{bmatrix} \in [\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}] \ominus \text{span} \left\{ \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\},$$

then

$$0 = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \right\rangle_{\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}} = \langle f, \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \rangle_{\mathcal{H}_{K_1}} + \langle g, \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \rangle_{\mathcal{H}_{K_2}},$$

that is, $w_1 f(\boldsymbol{w}) + w_2 g(\boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathbb{D}^2$. Since K_1 depends only on z_1 and $\bar{w}_1$, all the functions in $\mathcal{H}_{K_1}$ depends only on z_1. Therefore, if $w_2 = 0$, then the above equality implies that $w_1 f((w_1, 0)) = 0$, and hence $f = 0$. Consequently, $w_2 g(\boldsymbol{w}) = 0$, $\boldsymbol{w} \in \mathbb{D}^2$, and hence $g = 0$, and proves our claim. In particular, $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2})$ is an isometry. The above proof also implies that

$$\mathcal{H}_{K_i} = \overline{\text{span}} \{ \bar{w}_i K_i(\cdot, \boldsymbol{w}) : \boldsymbol{w} \in \mathbb{D}^2 \},$$

for $i = 1, 2$. Now we consider the co-isometry V^* and set

$$V^* = \begin{bmatrix} \varphi(\mathbf{0}) & B \\ C & D \end{bmatrix} = \begin{bmatrix} \varphi(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2})).$$

Since

$$\begin{bmatrix} \overline{\varphi(\mathbf{0})} & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

it follows that

$$\overline{\varphi(\mathbf{0})} + C^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \overline{\varphi(\boldsymbol{w})},$$

and

$$B^* + D^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix},$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Now plug $\boldsymbol{w} = \boldsymbol{0}$ into the identity above to see that

$$B^* = \begin{bmatrix} K_1(\cdot, \boldsymbol{0}) \\ K_2(\cdot, \boldsymbol{0}) \end{bmatrix},$$

and hence

$$D^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}) \\ K_2(\cdot, \boldsymbol{w}) - K_2(\cdot, \boldsymbol{0}) \end{bmatrix}.$$

Since $D^* = \begin{bmatrix} D_1^* & D_3^* \\ D_2^* & D_4^* \end{bmatrix}$, it follows that

$$\bar{w}_1 D_1^* K_1(\cdot, \boldsymbol{w}) + \bar{w}_2 D_3^* K_2(\cdot, \boldsymbol{w}) = K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}),$$

and

$$\bar{w}_1 D_2^* K_1(\cdot, \boldsymbol{w}) + \bar{w}_2 D_4^* K_2(\cdot, \boldsymbol{w}) = K_2(\cdot, \boldsymbol{w}) - K_2(\cdot, \boldsymbol{0}). \tag{4.2}$$

Plugging $w_2 = 0$ into the first identity, we get

$$\bar{w}_1 D_1^* K_1(\cdot, (w_1, 0)) = K_1(\cdot, (w_1, 0)) - K_1(\cdot, \boldsymbol{0}),$$

for all $w_1 \in \mathbb{D}$. Again, noting that K_1 depends only on z_1 and $\bar{w}_1$, we deduce

$$\bar{w}_1 D_1^* K_1(\cdot, \boldsymbol{w}) = K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}) \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

and consequently $D_3^*\Big(\bar{w}_2 K_2(\cdot, \boldsymbol{w})\Big) = 0$, $\boldsymbol{w} \in \mathbb{D}^2$. This, along with the fact that $\{\bar{w}_2 K_2(\cdot, \boldsymbol{w}) : \boldsymbol{w} \in \mathbb{D}^2\}$ is dense in $\mathcal{H}_{K_2}$, implies $D_3 = 0$. We next plug $w_2 = 0$ into (4.2) to get

$$D_2^*(\bar{w}_1 K_1(\cdot, (w_1, 0))) = K_2(\cdot, (w_1, 0)) - K_2(\cdot, \boldsymbol{0}).$$

Now we turn to compute C_1^*. Since $C^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w} K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \overline{\varphi(\boldsymbol{w})} - \overline{\varphi(\boldsymbol{0})}$, we have

$$C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) + C_2^*(\bar{w}_2 K_2(\cdot, \boldsymbol{w})) = \overline{\varphi(\boldsymbol{w})} - \overline{\varphi(\boldsymbol{0})} \qquad (\boldsymbol{w} \in \mathbb{D}^2).$$

In particular, if $w_2 = 0$, then

$$C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = \overline{\varphi((w_1, 0))} - \overline{\varphi(\boldsymbol{0})} \qquad (w_1 \in \mathbb{D}).$$

Finally, we compute B_2. Observe that

$$B_2^* + D_2^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) + D_4^*(\bar{w}_2 K_2(\cdot, \boldsymbol{w})) = K_2(\cdot, \boldsymbol{w}),$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. If $w_2 = 0$, then

$$B_2^* + D_2^*(\bar{w}_1 K_1(\cdot, (w_1, 0))) = K_2(\cdot, (w_1, 0)),$$

which implies that $B_2^* = K_2(\cdot, \boldsymbol{0})$. Finally, if we let $\boldsymbol{w} \in \mathbb{D}^2$, then

$$B_2^* C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = (\overline{\varphi(w_1, 0)} - \overline{\varphi(\boldsymbol{0})}) K_2(\cdot, \boldsymbol{0}) = \overline{\varphi(\boldsymbol{0})} K_2(\cdot, (w_1, 0)) - \overline{\varphi(\boldsymbol{0})} K_2(\cdot, \boldsymbol{0}),$$

by assumption (4.1), and hence

$$B_2^* C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = \overline{\varphi(\mathbf{0})}(K_2(\cdot, (w_1, 0)) - K_2(\cdot, \mathbf{0})) = \overline{\varphi(\mathbf{0})} D_2^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})).$$

This proves that $\varphi(\mathbf{0}) D_2 = C_1 B_2$.

(4)$\Rightarrow$ (1) is essentially along the lines of [10, Theorem 2.3]. However, for the sake of completeness, we sketch the proof. Let $a = \varphi(\mathbf{0})$. Since $VV^* = I$, it follows that

$$I = \begin{bmatrix} |a|^2 + B_1 B_1^* + B_2 B_2^* & aC_1^* + B_1 D_1^* + B_2 D_2^* & aC_2^* + B_2 D_4^* \\ \overline{a} C_1 + D_1 B_1^* + D_2 B_2^* & C_1 C_1^* + D_1 D_1^* + D_2 D_2^* & C_1 C_2^* + D_2 D_4^* \\ \overline{a} C_2 + D_4 B_2^* & C_2 C_1^* + D_4 D_2^* & C_2 C_2^* + D_4 D_4^* \end{bmatrix}.$$

Then there exists $y \in \mathbb{C}$ such that

$$|y|^2 = |a|^2 + B_2 B_2^* = 1 - B_1 B_1^* > 0,$$

as $a \neq 0$. Let $x = \frac{a}{y}$, and

$$V_1 = \begin{bmatrix} y & B_1 \\ \frac{1}{x} C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} x & \frac{1}{y} B_2 \\ C_2 & D_4 \end{bmatrix}.$$

Clearly, $x \neq 0$. We first claim that V_1 and V_2 are co-isometries. Indeed

$$V_2 V_2^* = \begin{bmatrix} |x|^2 + \frac{1}{|y|^2} B_2 B_2^* & x C_2^* + \frac{1}{y} B_2 D_4^* \\ \bar{x} C_2 + \frac{1}{\bar{y}} D_4 B_2^* & C_2 C_2^* + D_4 D_4^* \end{bmatrix} = \begin{bmatrix} 1 & x C_2^* + \frac{1}{y} B_2 D_4^* \\ \bar{x} C_2 + \frac{1}{\bar{y}} D_4 B_2^* & C_2 C_2^* + D_4 D_4^* \end{bmatrix}.$$

as $|y|^2 = |a|^2 + B_2 B_2^*$ and $a = xy$. Also note that, since $aC_2^* + B_2 D_4^* = 0$, we have that $x C_2^* + \frac{1}{y} B_2 D_4^* = 0$, which implies that V_2 is a co-isometry. Next, we compute

$$V_1 V_1^* = \begin{bmatrix} |y|^2 + B_1 B_1^* & \frac{y}{x} C_1^* + B_1 D_1^* \\ \frac{\bar{y}}{x} C_1 + D_1 B_1^* & \frac{1}{|x|^2} C_1 C_1^* + D_1 D_1^* \end{bmatrix}.$$

Since $C_1 C_1^* + D_1 D_1^* + D_2 D_2^* = 1$, $aD_2 = C_1 B_2$, $a = xy$ and $|y|^2 - |a|^2 = B_2 B_2^*$, we have

$$\frac{1}{|x|^2} C_1 C_1^* + D_1 D_1^* = 1.$$

Moreover, since $aC_1^* + B_1 D_1^* + B_2 D_2^* = 0$ implies that $\frac{y}{x} C_1^* + B_1 D_1^* = 0$, we have that V_1 is also a co-isometry. Finally, set $\varphi_1(\boldsymbol{z}) = \tau_{V_1}(z_1)$ and $\varphi_2(\boldsymbol{z}) = \tau_{V_2}(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$. It is then easy to check that

$$\varphi(\boldsymbol{z}) = \tau_V(z) = \tau_{V_1}(z_1) \tau_{V_2}(z_2) = \varphi_1(\boldsymbol{z}) \varphi_2(\boldsymbol{z}),$$

for all $\boldsymbol{z} \in \mathbb{D}^2$. This completes the proof. ∎

In the setting of Theorem 4.1, one can also explicitly compute the entries of the block operator matrix V in part (4). The technique involved in the computation is standard and quite well known (cf. [5, Remark 3.6]). However, we outline some details for the sake of making this paper self-contained. We already know that

$$B_2^* = K_2(\cdot, \mathbf{0}) \quad \text{and} \quad C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = \overline{\varphi((w_1, 0))} - \overline{\varphi(\mathbf{0})},$$

and

$$D_2^*(\bar{w}_1 K_1(\cdot, (w_1, 0))) = K_2(\cdot, (w_1, 0)) - K_2(\cdot, \mathbf{0}),$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Now let $g \in \mathcal{H}_{K_2}$ and $\boldsymbol{w} \in \mathbb{D}^2$. Then

$$(z_1 D_2 g)(\boldsymbol{w}) = \langle g, K_2(\cdot,(w_1,0)) - K_2(\cdot,\mathbf{0})\rangle = g((w_1,0)) - g(\mathbf{0}),$$

and hence

$$(D_2 g)(\boldsymbol{w}) = \frac{g((w_1,0)) - g(\mathbf{0})}{w_1} \qquad (\boldsymbol{w} \in \mathbb{D}^2).$$

for all $g \in \mathcal{H}_{K_2}$. Similarly, if $w_1 = 0$, then (4.2) implies that

$$\bar{w}_2 D_4^* K_2(\cdot,\boldsymbol{w}) = K_2(\cdot,\boldsymbol{w}) - K_2(\cdot,(w_1,0)),$$

and hence, in a similar way we have

$$(D_4 g)(\boldsymbol{w}) = \frac{g(\boldsymbol{w}) - g((w_1,0))}{w_2} \qquad (g \in \mathcal{H}_{K_2}, \boldsymbol{w} \in \mathbb{D}^2),$$

as well as

$$(D_1 f)(\boldsymbol{w}) = \frac{f(\boldsymbol{w}) - f(\mathbf{0})}{w_1} \qquad (f \in \mathcal{H}_{K_1}, \boldsymbol{w} \in \mathbb{D}^2).$$

Now we turn to compute C_1 and C_2. Since $C_1^*(\bar{w}_1 K_1(\cdot,\boldsymbol{w})) = \overline{\varphi((w_1,0))} - \overline{\varphi(\mathbf{0})}$, we have

$$(z_1 C_1 1)(\boldsymbol{w}) = \langle C_1 1, \bar{w}_1 K_1(\cdot,\boldsymbol{w})\rangle = \varphi((w_1,0)) - \varphi(\mathbf{0}),$$

and hence

$$(C_1 1)(\boldsymbol{w}) = \frac{\varphi(w_1,0) - \varphi(\mathbf{0})}{w_1} \quad \text{and} \quad (C_2 1)(\boldsymbol{w}) = \frac{\varphi(\boldsymbol{w}) - \varphi(w_1,0)}{w_2},$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Finally, we note that $(B_1 f)(\boldsymbol{w}) = f(\mathbf{0})$ and $(B_2 g)(\boldsymbol{w}) = g(\mathbf{0})$ for all $f \in \mathcal{H}_{K_1}$ and $g \in \mathcal{H}_{K_2}$.

In particular, if φ is inner, then we have the following:

Example 4.2. Given an inner function $\varphi \in \mathcal{S}(\mathbb{D}^2)$ satisfying one of the equivalent conditions of Theorem 4.1, we have $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$. Then

$$1 = |\varphi(\boldsymbol{z})| = |\varphi_1(z_1)||\varphi_2(z_2)| \leq |\varphi_1(z_1)| \leq 1 \qquad (\boldsymbol{z} \in \mathbb{T}^2 \text{ a.e.})$$

from which we see that φ_1, as well as φ_2, are inner functions. Moreover, for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$, we have

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = 1 - \varphi_1(z_1)\overline{\varphi_1(w_1)} + \varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\overline{\varphi_1(w_1)}.$$

Hence $\{K_1, K_2\}$ are Agler kernels of φ, where

$$K_1(\boldsymbol{z},\boldsymbol{w}) = \frac{1 - \varphi_1(z_1)\overline{\varphi_2(w_1)}}{1 - z_1\bar{w}_1} \quad \text{and} \quad K_2(\boldsymbol{z},\boldsymbol{w}) = \frac{\varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\overline{\varphi_1(w_1)}}{1 - z_2\bar{w}_2}.$$

In this case the corresponding reproducing kernel Hilbert spaces are given by

$$\mathcal{H}_{K_1} = \mathcal{Q}_{\varphi_1} \otimes \mathbb{C} \quad \text{and} \quad \mathcal{H}_{K_2} = \varphi_1\mathbb{C} \otimes \mathcal{Q}_{\varphi_2},$$

where $\mathcal{Q}_{\varphi_1} = H^2(\mathbb{D})/\varphi_1 H^2(\mathbb{D})$ and $\mathcal{Q}_{\varphi_2} = H^2(\mathbb{D})/\varphi_2 H^2(\mathbb{D})$ are model spaces. Moreover, the co-isometric (unitary) colligation operator V with state space $\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$ is given by

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_1}} & \varphi_1(0) P_{\mathbb{C}} M_{\varphi_1}^* \otimes P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_2}} \\ \varphi_2(0) M_z^* M_{\varphi_1}|_{\mathbb{C}} & M_z^*|_{\mathcal{Q}_{\varphi_1}} & M_z^* M_{\varphi_1} P_{\mathbb{C}} M_{\varphi_1}^* \otimes P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_2}} \\ M_{\varphi_1}|_{\mathbb{C}} \otimes M_z^* M_{\varphi_2}|_{\mathbb{C}} & 0 & I_{\varphi_1 \mathbb{C}} \otimes M_z^*|_{\mathcal{Q}_{\varphi_2}} \end{bmatrix}.$$

Finally, we comment on the assumption that $\varphi(\mathbf{0}) \neq 0$ in Theorem 4.1.

Remark 4.3. In the proof of Theorem 4.1, $\varphi(\mathbf{0}) \neq 0$ has been used only for the implication (4)$\Rightarrow$ (1). In the $\varphi(\mathbf{0}) = 0$ case, one can easily modify the argument of the aforementioned case to prove a similar statement. Here is a sample statement:
Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ be a non-zero function and suppose $\varphi(\mathbf{0}) = 0$. Then the following are equivalent:

(1) $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$ for some $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{D})$ such that $\varphi_2(0) \neq 0$.

(2) $\varphi(\boldsymbol{z}) = z_1^p \varphi_1(z_1)\varphi_2(z_2)$ for some $p \geq 1$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{D})$ such that $\varphi_1(0) \neq 0$ and $\varphi_2(0) \neq 0$.

(3) There exists $p \geq 1$ such that $\tilde{\varphi}(\boldsymbol{z}) = z_1^{-p}\varphi(\boldsymbol{z}) \in \mathcal{S}(\mathbb{D}^2)$, $\tilde{\varphi}(\mathbf{0}) \neq 0$, and there exist Agler kernels $\{K_1, K_2\}$ of $\tilde{\varphi}$ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\tilde{\varphi}(\mathbf{0})} K_2(\cdot, (w_1, 0)) = \overline{\tilde{\varphi}(w_1, 0)} K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(4) There exists $p \geq 1$ such that $\tilde{\varphi}(\boldsymbol{z}) = z_1^{-p}\varphi(\boldsymbol{z}) \in \mathcal{S}(\mathbb{D}^2)$, $\tilde{\varphi}(\mathbf{0}) \neq 0$, and $\tilde{\varphi} = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \tilde{\varphi}(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix},$$

such that $\tilde{\varphi}(\mathbf{0}) D_2 = C_1 B_2$.

5. Counterexamples and a converse

We now return to two-variable inner functions, which we encountered in Section 2. The aim of this section is to further analyze Theorem 2.1. We begin by exhibiting counterexamples to the converse of Theorem 2.1. Then, in Theorem 5.3, we present a weak converse to Theorem 2.1.

Example 5.1. Fix $t \in (0, 1)$, and define

$$\varphi_t(\boldsymbol{z}) = \frac{z_1 z_2 - t}{1 - t z_1 z_2} \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

It is fairly easy to verify that

$$|\varphi_t(\boldsymbol{z})| = 1 \qquad (\boldsymbol{z} \in \mathbb{T}^2),$$

and hence, φ_t is a rational inner function. Contrary to what we want, let us assume that there are Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, an operator $D_1 \in C_{0\cdot}$, and an isometric colligation

$$V_t = \begin{bmatrix} -t & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$$

such that $\tau_{V_t} = \varphi_t$. Since
$$\varphi_t(\boldsymbol{z}) + t = \frac{(1-t^2)z_1z_2}{1-tz_1z_2},$$
the preceding equality yields
$$\frac{(1-t^2)z_1z_2}{1-tz_1z_2} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$
Now the left side is equal to
$$(1-t^2)z_1z_2(1 + tz_1z_2 + t^2z_1^2z_2^2 + \cdots),$$
and the right side is equal to
$$z_1B_1(I - z_1D_1)^{-1}C_1 + z_2B_2(I - z_2D_4)^{-1}C_2 + z_1z_2B_1(I - z_1D_1)^{-1}D_2(I - z_2D_3)^{-1}C_2.$$
Comparing the coefficients of z_1, we see that $B_1D_1^nC_1 = 0$, $n \geq 0$. Since $V_t^*V_t = I$, we have
$$\begin{bmatrix} -t & C_1^* & C_2^* \\ B_1^* & D_1^* & 0 \\ B_2^* & D_2^* & D_3^* \end{bmatrix} \begin{bmatrix} -t & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$
In particular $B_1^*B_1 + D_1^*D_1 = I$ and $-tB_1 + C_1^*D_1 = 0$. The first equality implies (see the proof of the equality in (2.4)) that
$$\sum_{n=0}^{\infty} D_1^{*n}B_1^*B_1D_1^n = I,$$
in the strong operator topology as $D_1 \in C_{0\cdot}$. Therefore
$$\sum_{n=0}^{\infty} \|B_1D_1^nh\|^2 = \|h\|^2,$$
for all $h \in \mathcal{H}_1$. In particular, if we choose $h = C_1(1)$, then
$$\sum_{n=0}^{\infty} \|B_1D_1^nC_1(1)\|^2 = \|C_1(1)\|^2.$$
Since $B_1D_1^nC_1 = 0$ for all $n \geq 0$, we deduce$C_1 = 0$. Then $-tB_1 + C_1^*D_1 = 0$ implies that $B_1 = 0$, and hence $D_1^*D_1 = I$. However, this and the fact that $D_1 \in C_{0\cdot}$ are mutually contradictory. This shows that $\varphi_t \neq \tau_{V_t}$ for any isometric colligation V_t and $D_1 \in C_{0\cdot}$.

Now we turn to a weak converse of Theorem 2.1 in the setting of rational inner functions. We need the following inverse formula of 2×2 block matrices [14, page 18]:

Theorem 5.2. *Let* $X = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathcal{B}(\mathbb{C}^m \oplus \mathbb{C}^n)$*, and suppose that* P *is invertible. Then* X *is invertible if and only if* $\Delta := S - RP^{-1}Q$ *is invertible. In this case, the inverse of* X *is given by*
$$X^{-1} = \begin{bmatrix} P^{-1} + P^{-1}Q\Delta^{-1}RP^{-1} & -P^{-1}Q\Delta^{-1} \\ -\Delta^{-1}RP^{-1} & \Delta^{-1} \end{bmatrix}.$$

We are now ready to establish the promised weak converse of Theorem 2.1.

Theorem 5.3. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ be a rational inner function and suppose $\varphi(\mathbf{0}) \neq 0$. Then the following are equivalent:*
(1) $\varphi = \tau_V$ for some isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional Hilbert spaces and $D_1, D_3 \in C_{0\cdot}$.
(2) $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some rational inner functions φ_1 and φ_2 (in $\mathcal{S}(\mathbb{D})$).

Proof. (1)$\Rightarrow$ (2): Since $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$ is an isometry and $\dim \mathcal{H}_i < \infty$, $i = 1, 2$, V is onto, that is, V is a unitary operator. In particular, V is invertible. Since

$$a = \varphi(\mathbf{0}) \neq 0,$$

by the above theorem, we conclude that $aD - CB$ is invertible and

$$V^{-1} = \begin{bmatrix} a^{-1} + a^{-1}B(aD - CB)^{-1}C & -B(aD - CB)^{-1} \\ -(aD - CB)^{-1}C & a^{-1}(aD - CB)^{-1} \end{bmatrix}.$$

Since $V^* = V^{-1}$, in particular, we have

$$D^* = \begin{bmatrix} D_1^* & 0 \\ D_2^* & D_3^* \end{bmatrix} = a^{-1}(aD - CB)^{-1} = a^{-1} \begin{bmatrix} aD_1 - C_1B_1 & aD_2 - C_1B_2 \\ -C_2B_1 & aD_3 - C_2B_2 \end{bmatrix}^{-1},$$

and hence

$$a \begin{bmatrix} aD_1 - C_1B_1 & aD_2 - C_1B_2 \\ -C_2B_1 & aD_3 - C_2B_2 \end{bmatrix} \begin{bmatrix} D_1^* & 0 \\ D_2^* & D_3^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

But then this implies $(aD_2 - C_1B_2)D_3^* = 0$. Note that the invertibility of D immediately implies that D_3 is also invertible. Then $aD_2 - C_1B_2 = 0$, and hence, by Theorem 4.1, there exist rational inner functions φ_1 and φ_2 (here $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional Hilbert spaces) such that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$.
(2)$\Rightarrow$ (1): Since φ_i ($\in \mathcal{S}(\mathbb{D})$) is a rational inner function, there exists an isometric colligation

$$V_i = \begin{bmatrix} a_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_i),$$

such that $\dim(\mathcal{H}_i) < \infty$, $D_i \in C_{0\cdot}$, and $\varphi_i = \tau_{V_i}$ for all $i = 1, 2$. We define

$$V = \begin{bmatrix} a_1a_2 & B_1 & a_1B_2 \\ a_2C_1 & D_1 & C_1B_2 \\ C_2 & 0 & D_2 \end{bmatrix}.$$

Then a somewhat careful computation (or see the proof of [10, Theorem 2.2]) yields that $\varphi = \tau_V$. ∎

In this connection, and also in the context of Remark 2.2, it is probably worth mentioning that in the finite dimensional case we have the following: If $\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C}^p \oplus \mathbb{C}^q)$ for some $p, q \geq 1$, then

$$\sigma\Big(\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \Big) = \sigma(D_1) \cup \sigma(D_3),$$

and in particular, $\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in C_{0\cdot}$ if and only if $D_1, D_3 \in C_{0\cdot}$.

Finally, we point out that part (1) of Theorem 4.1 and part (2) of Theorem 5.3 are related (in a different direction) to essential normality of Beurling type quotient modules of $H^2(\mathbb{D}^2)$ [13].

Acknowledgement: The second author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014, and the Mathematical Research Impact Centric Support (MATRICS) grant, File No : MTR/2017/000522, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India.

References

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, Topics in operator theory: Ernst D. Hellinger memorial volume, Oper. Theory Adv. Appl., vol. 48, Birkhäuser, Basel, 1990, 47-66.

[2] J. Agler, *Some interpolation theorems of Nevanlinna-Pick type*, (1988, preprint).

[3] J. Agler and J. McCarthy, *Nevanlinna-Pick interpolation on the bidisk*, J. reine angew. Math. 506 (1999), 191–204.

[4] J. Agler, J. McCarthy and N. Young, *Operator Analysis: Hilbert Space Methods in Complex Analysis*, Cambridge Tracts in Mathematics, 219. Cambridge University Press, 2020.

[5] J. Ball and V. Bolotnikov, *Canonical de Branges-Rovnyak model transfer-function realization for multivariable Schur-class functions*, Hilbert spaces of analytic functions, CRM Proc. Lecture Notes, 51, Amer. Math. Soc., Providence, RI, 2010, 1–39.

[6] K. Bickel, J.E. Pascoe, and R. Tully-Doyle *Analytic continuation of concrete realizations and the McCarthy Champagne conjecture*, Preprint available at https://arxiv.org/abs/2009.14188.

[7] K. Bickel and C. Liaw, *Properties of Beurling-type submodules via Agler decompositions*, J. Funct. Anal. 272 (2017), 83-111.

[8] L. de Branges and J. Rovnyak, *The existence of invariant subspaces*, Bull. Amer. Math. Soc. 70 (1964), 718-721.

[9] L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory.* 1966 Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965) pp. 295–392 Wiley, New York.

[10] R. Debnath and J. Sarkar, *Factorizations of Schur functions*, Complex Anal. Oper. Theory 15 (2021), Paper No. 49, 31 pp.

[11] C. Foias and A. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhäuser Verlag, Basel, 1990.

[12] C. Foias, A. Frazho, I. Gohberg and M. Kaashoek, *Metric constrained interpolation, commutant lifting and systems*, Operator Theory: Advances and Applications, 100. Birkhäuser Verlag, Basel, 1998.

[13] K. Guo and K. Wang, *Beurling type quotient modules over the bidisk and boundary representations*, J. Funct. Anal. 257 (2009), 3218–3238.

[14] R. Horn and C. Johnson, Matrix analysis. *Cambridge University Press*, Cambridge, 1985.
[15] G. Knese, *Kummert's Approach to Realization on the Bidisk*, https://arxiv.org/pdf/1907.13191.pdf.
[16] A. Maji, J. Sarkar and Sankar T. R. *Pairs of commuting isometries, I*, Studia Math. 248 (2019), 171–189.
[17] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc., New York–Amsterdam, 1969.
[18] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.
[19] V. Paulsen and M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, 152. Cambridge University Press, Cambridge, 2016.
[20] F. H. Szafraniec, *The reproducing kernel Hilbert space and its multiplication operators*, Complex analysis and related topics (Cuernavaca, 1996), 253–263, Oper. Theory Adv. Appl., 114, Birkh¨user, Basel, 2000.
[21] H. Woerdeman, *A general Christoffel–Darboux type formula*, Integral Equations Operator Theory 67 (2) (2010) 203–213.
[22] S. Zhu, Y. Yang and Y. Lu, *The reducibility of compressed shifts on Beurling type quotient modules over the bidisk.* J. Funct. Anal. 278 (2020), 108304.

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
Email address: ramlal_rs@isibang.ac.in, ramlaldebnath100@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
Email address: jay@isibang.ac.in, jaydeb@gmail.com